Seat No. : _______________

JD-109

January-2018

M.Sc., Sem.-I

405 : Mathematics

(Measure and Integration)

(Old Course)

Time : 3 Hours] **[Max. Marks : 70**

1. (a) Attempt any **one** : **7**

(1) Prove that a subset E of [a, b] is measurable if and only if for given $\varepsilon > 0$, there exists open sets G_1 and G_2 such that $G_1 \supseteq E$, $G_2 \supseteq E'$ and $|G_1 \cap G_2| < \varepsilon$.

(2) If $E \subseteq [0, 1]$, then show that $\overline{m}E + \underline{m}E' = 1$. Further, deduce that E is measurable if and only if $\overline{m}E + \overline{m}E' \leq 1$.

(b) Attempt any **two** : **4**

(1) Let $E \subset [a, b]$ be measurable. Prove that there exists a closed set $F \subseteq E$ and an open set $G \supseteq E$ such that $|G| - |F| < \frac{1}{2}$.

(2) Prove that every open subset of [a, b] is measurable.

(3) Let $E = [0, 1] \cup \{-2, 2, 4\} \subseteq [-5, 4]$. Determine $\overline{m}E$ by the definition of outer measure.

(c) Answer in brief : **3**

(1) Give the definition of length for a closed subset of [a, b].

(2) Give the definition of outer measure for a subset E of [a, b].

(3) True or False : If mE = 1, then E is singleton set.

2. (a) Attempt any **one** : **7**

(1) State and prove the Caratheodary test for measurability.

(2) If E_1 and E_2 are subsets of [a, b] then prove that $\overline{m}E_1 + \overline{m}E_2 \geq \overline{m}(E_1 \cup E_2) + \overline{m}(E_1 \cap E_2)$ and $\underline{m}E_1 + \underline{m}E_2 \leq \underline{m}(E_1 \cup E_2) + \underline{m}(E_1 \cap E_2)$.

(b) Attempt any **two** : **4**

(1) Let $E = [0, 1] \cap \mathbb{Q}$. Show that the function $f(x) = \chi_E(x)$ is measurable on $[0, 1]$ by definition.

(2) If $f : [a, b] \to (0, \infty)$ is measurable then show that the function $\frac{1}{f}$ is also measurable on $[a, b]$.

(3) If E_1 and E_2 are measurable sets and $E_2 \subset E_1$, then show that $E_1 - E_2$ is measurable and $m(E_1 - E_2) = mE_1 - mE_2$.

(c) Answer in brief : **3**

(1) If $mA = 0$ and $B = \{1\}$; what is the measure of $A \cup B$?

(2) Define : Isometry.

(3) True or False : Every measurable function is continuous on $[a, b]$.

3. (a) Attempt any **one** : **7**

(1) If f and g are bounded measurable functions in $L[a, b]$, then prove that $f + g$ is in $L[a, b]$ and $\int_a^b f + g = \int_a^b f + \int_a^b g$.

(2) Prove that every bounded measurable function on $[a, b]$ is Lebesgue integrable on $[a, b]$.

(b) Attempt any **two** : **4**

(1) Prove or disprove : If f is a non-negative bounded measurable function on $[a, b]$ such that $\int_a^b f = 0$, then $f = 0$, a.e. in $[a, b]$.

(2) Let E_1 and E_2 denote the set of rationals and irrationals is $[0, 1]$ respectively. If $= \chi_{E_1} + \chi_{E_2}$, then compute $\int_0^1 f$.

(3) Let $E_1 = [0, 1] \cap \mathbb{Q}$, $E_2 = [0, 1] - E_1$ and $P = \{E_1, E_2\}$ be a measurable partition of $[0, 1]$. If $f(x) = x^2$ on $[0, 1]$ then compute $u(f, P)$.

(c) Answer in brief : **3**

(1) If $f(x) = \sin x$ and $E = \left\{\frac{1}{n} : n \in N\right\}$, then what is the value of $\int_E f$?

(2) Give the definition of measurable partition of [a, b].

(3) True or false : If $\int_E f = 0$, then $f = 0$, a.e. in E.

4. (a) Attempt any **one** : **7**

(1) State and prove the Lebesgue's dominated convergence theorem.

(2) State and prove Fatou's Lemma and deduce the monotone convergence theorem.

(b) Attempt any **two** : **4**

(1) What we mean absolute continuity of the Lebesgue integral ?

(2) If a measurable function $f \in L[a, b]$ and $\lambda \in R$, then show that

$$\int_a^b \lambda f = \lambda \int_a^b f.$$

(3) If $f(x) = 1 - \cos x$, $0 \le x \le \pi$, then find f^+ and f^-.

(c) Answer in brief : **3**

(1) Explain what we mean by the countable additivity of the Lebesgue integral.

(2) Let f be a measurable function on [a, b]. Show that $f \in L[a, b]$ if and only if $|f| \in L[a, b]$.

(3) True or false : $f \in L[0, 1] \Rightarrow f^2 \in L[0, 1]$.

5. (a) Attempt any **one** : **7**

(1) Prove that the metric space $L_2[a, b]$ is complete.

(2) Determine the Fourier series of the function $f(x) = 3x^2$.

(b) Attempt any **two** : **4**

(1) State and prove Schwarz's inequality.

(2) State and prove Minkowski's inequality.

(3) Determine the Fourier series of the function $f(x) = \sin x \cos x + \sin^2 x - \cos^2 3x$.

(c) Answer in brief : **3**

(1) When do we say that a trigonometric series is a Fourier series ?

(2) Show that $C[a, b] \subseteq L^2[a, b]$.

(3) True or false : If $f, g \in L_2[a, b]$, then $fg \in L[a, b]$.